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ON THE EXISTENCE OF SEPARATRIX LOOPS IN FOUR-DIMENSIONAL SYSTEMS SIMILAR TO THE INTEGRABLE HAMILTONIAN SYSTEMS*

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A method analogous to the V.K. Mel'nikov method /1/ is used to derive the conditions of existence of separatrix loops of the saddle-focus type singularity, for the systems similar to the integrable Hamiltonian systems. Many important and interesting effects appearing in the dynamic systems are connected with the presence of loops in the separatrices of the singularities. For example, a system with a loop of a saddle-focus separatrix with the positive saddle value has an enumerable set of periodic saddle-type motions /2/; the appearance of the saddle separatrix loops can represent the first of a sequence of bifurcations leading to the appearance of an attractor /3/. The existence of a separatrix loop in a finite-dimensional system describing the travelling wave-type solutions of partial differential equations implies the presence of a soliton solution in the latter system /4/.

1. Let us consider in R^4 the Hamiltonian system

$$x^{\cdot} = I\nabla H_1(x) \tag{1.1}$$

with a smooth Hamiltonian $H_1(x)$ where I is a (4×4) matrix in which $a_{31}=a_{42}=-a_{13}=-a_{24}=1$ and the remaining elements are equal zero, and $\nabla H_1(x)$ is the gradient vector of the function $H_1(x)$. In what follows, when we say smooth, we mean the C^∞-smoothness unless otherwise stated. We assume that the system (1.1) is integrable, i.e. a smooth function $H_2(x)$ exists, representing the first integral of the system. This means that the Poisson's bracket $\{H_1, H_2\}\equiv(I\nabla H_1, \nabla H_2)$ of the functions H_1 and H_2 is equal to zero, where $(\cdot,\cdot)$ denotes the standard scalar product in R^4. Sometimes we shall consider, together with the system (1.1), a Hamiltonian system with the Hamiltonian $H_2(x)$. We assume that the above systems generate dynamic systems, i.e. every trajectory is continued onto all $t\in R^1$. A sufficient condition for this is e.g. the condition that the levels of one of the integrals be compact subsets in R^4.

Let the coordinate origin O be a singularity of the system (1.1), the roots of the characteristic equation of which are $\pm\alpha_1\pm i\beta_1$, $\alpha_1\cdot\beta_1\neq 0$ (the singularity is of the saddle-focus type). From the theorem on a stable manifold /5/ it follows that smooth, two-dimensional stable W_0^s and unstable W_0^u invariant manifolds of the singularity O exist. The functions $H_1(x)$ and $H_2(x)$ can be simultaneously reduced by a linear variable change to the form (old notation is retained for the new coordinates)

$$H_k(x)=H_k(0)+\alpha_k(p_1q_1+p_2q_2)-\beta_k(p_1q_2-p_2q_1)+F_k(x),\quad F_k(x)=o(\|x\|^3);\quad k=1,2$$

In what follows, we assume:

A) the quantity $\tau=\alpha_1\beta_2-\alpha_2\beta_1\neq 0$;

B) a trajectory $\gamma\subset W_0^s\cap W_0^u$ exists doubly asymptotic to O.

Condition A) means that H_1 and H_2 are independent in some neighborhood U of the point O (except, of course, for the point O itself), i.e. the vectors $I\nabla H_1(x)$ and $I\nabla H_2(x)$ are linearly independent (**). We shall call such a singularity nondegenerate.

Proposition 1. Let conditions A) and B) hold. Then the closures of the manifolds W_0^s and W_0^u coincide, i.e. all trajectories in $W_0^s-\{O\}$ (and in $W_0^u-\{O\}$) are doubly asymptotic to O.

Proof. Let $f(t;x)$ be a trajectory of the system (1.1) passing through the point x at $t=0$, and let $g(s;x)$ be the analogous trajectory of the system with the Hamiltonian H_2.

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**) Lerman L.M. and Umanskii Ia.L. Structure of the Poissonian action R^2 on the four-dimensional symplectic manifold. I. Gor'kii, Dep. v VINITI 1981.

Since $\{H_1, H_2\} \equiv 0$, we have $f(t; g(s; x)) \equiv g(s; f(t; x))$ [6]. Therefore we can determine the action of the group R^2 on the phase space R^4 using the formula $\Phi(t, s; x) = f(t; g(s; x)), (t, s) \in R^2$. Any two orbits of the action either do not intersect, or they coincide. As was shown before in (**), the manifolds $W_0^s - \{0\}$ and $W_0^u - \{0\}$ are two-dimensional orbits of the action Φ. The existence of the doubly asymptotic trajectory γ implies that the above two-dimensional orbits intersect, therefore they must coincide, and this completes the proof. Thus when conditions A) and B) hold, a one-parameter family of trajectories doubly asymptotic to the point O exists in the system (1.1). We denote the set of points R^4 through which the trajectories of the family pass, by W_0.

2. Let us consider a perturbed system

$$x^{\cdot} = I\nabla H_1(x) + p(x, \varepsilon), \ \varepsilon \in R^m \tag{2.1}$$

where $p(x, \varepsilon)$ is a function C^k-smooth in ε $(k \geqslant 2)$ and smooth in x, $p(x, 0) \equiv 0$. We shall also assume, without loss of generality, that $p(0, \varepsilon) \equiv 0$. We construct in the neighborhood U of the point O a secant to the trajectory of the system (2.1). Restriction of (1.1) to W_0^s has a coarse focus-type singularity, therefore a smooth closed curve σ can be chosen on W_0^s, transverse to the trajectories. Let $x_0(\theta)$ be the parametric description of the curve σ $(0 \leqslant \theta \leqslant 2\pi)$. We take, at every point of $x_0(\theta)$, a two-dimensional plane N_θ stretched over the vectors $\nabla H_j(x_0(\theta))$ $(j = 1, 2)$. The plane N_θ is orthogonal to the plane tangent to the manifold W_0^s at the point $x_0(\theta)$, since the tangent plane is generated by the vectors $I\nabla H_1(x_0(\theta))$ and $I\nabla H_2(x_0(\theta))$. Using this fact we can show that $\mu > 0$ exists such, that the union (over θ) of the two-dimensional discs $\| x - x_0(\theta) \| < \mu$ belonging to N_θ forms a three-dimensional secant N homeomorphic to the full torus $R^2 \times S^1$. Clearly, N remains a secant for the system (2.1) at sufficiently small $\| \varepsilon \|$.

When $\varepsilon = 0$, the trace of W_0^s on N is represented by the curve σ, which is also the trace of W_0^u on N. Since the stable manifold of the coarse equilibrium state depends smoothly on the parameter, it follows that when $\varepsilon \neq 0$, then the stable manifold W_ε^s of the singularity O intersects transversally with N, and this intersection is therefore a smooth closed curve σ_ε^s close to σ. The smooth curve $\sigma_\varepsilon^u = W_\varepsilon^u \cap N$ is determined in the same manner. The loops of the separatrices correspond to the general points of the curves σ_ε^s and σ_ε^u. From this it is clear that, generally speaking, the curves σ_ε^s and σ_ε^u do not intersect at fixed $\varepsilon \neq 0$. However, if instead of an individual perturbed system we consider an at least two-parameter family of perturbations, then curves may exist in the parameter space with points to which systems with separatrix loops correspond.

For the Hamiltonian perturbations $p(x, \varepsilon) = I\nabla K(x, \varepsilon)$ the manifolds W_ε^s and W_ε^u lie on the set Σ_ε defined by the equation $H_1(x) + K(x, \varepsilon) = H_1(0) + K(0, \varepsilon)$. In the neighborhood of the set W_0, Σ_ε is a small, three-dimensional submanifold everywhere except at the point O, and the intersection of Σ_ε and N is homeomorphic to a two-dimensional ring. The curves σ_ε^s and σ_ε^u lie in this ring, and for this reason it is sufficient to consider a one-parameter family of perturbations. Below we shall discuss both cases.

3. Let us consider a perturbed system of the form (2.1), where $\varepsilon = (\varepsilon_1, \varepsilon_2)$ belongs to the neighborhood of the zero in R^2. We denote by $x_\varepsilon^s(t, \theta)$ the trajectory belonging to W_ε^s and intersecting, at $t = 0$, the plane N_θ. Such a trajectory exists and is unique. We determine the trajectory $x_\varepsilon^u(t, \theta)$ in the same manner. The functions $x_\varepsilon^s(t, \theta)$, $x_\varepsilon^u(t, \theta)$ are smooth in t and θ and C^k-smooth in ε, and can therefore be written in the form

$$x_\varepsilon^\varkappa(t, \theta) = x_0(t, \theta) + \sum_{|n|=1}^{k} \varepsilon^n \xi_n^\varkappa(t, \theta) + a_\varkappa(t, \theta, \varepsilon); \ \varkappa = s, u; \ n = (l, m) \tag{3.1}$$

$$\varepsilon^n = \varepsilon_1^l \varepsilon_2^m, \ |n| = l + m, \ \xi_n^\varkappa(t, \theta) = \xi_{l,m}^\varkappa(t, \theta) \ a_\varkappa(0, \theta, \varepsilon) = o(\| \varepsilon \|^k)$$

where $x_0(t, \theta)$ is a trajectory of the system (1) lying on W_0^s and intersecting N_θ at $t = 0$ (at the point $x_0(\theta)$), and l and m are nonnegative integers.

Lemma 1. The vector function $\xi_n^\varkappa(t, \theta)$ $(\varkappa = s, u)$ is a solution of the following system of differential equations:

$$\xi^{\cdot} = Id^2H_1(x_0(t, \theta))\xi + P_n(x_0(t, \theta), \xi_{10}^\varkappa(t, \theta), \ldots, \xi_{ij}^\varkappa(t, \theta), \ldots, \xi_{0,|n|-1}^\varkappa(t, \theta)) \tag{3.2}$$

where $d^2H_1(x)$ is the matrix of the second derivatives of the function $H_1(x)$, $i + j \leqslant |n| - 1$ and the smooth vector function P_n is determined by the right-hand part of the system (2.1). The solution satisfies the conditions 1) $\xi_n^\varkappa(0, \theta) \in N_\theta$; 2) $\xi_n^s(t, \theta) \to 0$ as $t \to \infty$, $\xi_n^u(t, \theta) \to 0$ as $t \to -\infty$, and is unique.

The proof of this lemma is analogous to that of Theorem 4 in /1/, therefore we shall only give it outline. Equations (3.1) for $\xi_n^\varkappa(t, \theta)$ are obtained by substituting (3.1) into (2.1), differentiating the resulting identity l times with respect to ε_1 and m times with respect to ε_2, and setting $\varepsilon_1=\varepsilon_2=0$ in the resulting identity. Condition 1) is obtained from (3.1) by setting $t=0$ and taking due regard to the fact that $x_\varepsilon^\varkappa(0,\theta)$ and $x_0(0,0)$ belong to N_θ at all ε. Condition 2) follows from the usual estimates for the solutions lying on a stable manifold /7/. The uniqueness of the solution with conditions 1) and 2) follows from the fact that system (3.2) possesses an exponential dichotomy of solutions.

Let us introduce the functions

$$\Delta_i(\theta,\ \varepsilon)=(\nabla H_i(x_0(\theta)),\ x_\varepsilon^s(\theta)-x_\varepsilon^u(\theta))\ (i=1,2),\ x_\varepsilon^\varkappa(\theta)\equiv x_\varepsilon^\varkappa(0,\ \theta)$$

The vector $x_\varepsilon^s(\theta)-x_\varepsilon^u(\theta)$ clearly belongs to N_θ. Therefore, by virtue of the linear independence of the vectors $\nabla H_i(x_0(\theta))$ $(i=1,2)$, $\Delta_1^2+\Delta_2^2=0$ if and only if $x_\varepsilon^s(\theta)=x_\varepsilon^u(\theta)$, i.e. the zeros of the system of equations $\Delta_1(\theta,\ \varepsilon)=0,\ \Delta_2(\theta,\ \varepsilon)=0$ are in one-to-one correspondence with the single circuit loops of the separatrices of the singularity O.

Let us obtain explicit expressions for $\Delta_1(\theta,\varepsilon)$ and $\Delta_2(\theta,\varepsilon)$ from the right-hand parts of the system (2.1). To do this we consider the functions $D_i(t,\ \theta,\ \varepsilon)=(\nabla H_i(x_0(t,\ \theta)),\ x_\varepsilon^s(t,\ \theta)-x_\varepsilon^u(t,\ \theta))$. Clearly, $D_i(0,\ \theta,\ \varepsilon)=\Delta_i(\theta,\ \varepsilon)$. Below we shall write for simplicity d^2H_i, ∇H_i to denote $d^2H_i(x_0(t,\ \theta))$, $\nabla H_i(x_0(t,\ \theta))$ respectively, and

$$P_n^\varkappa=P_n(x_0(t,\ \theta),\ \xi_{10}^\varkappa(t,\ \theta),\ldots,\ \xi_{1j}^\varkappa(t,\ \theta),\ldots,\ \xi_{0,|n|-1}^\varkappa(t,\ \theta))$$

Then

$$D_i(t,\ \theta,\ \varepsilon)=(\nabla H_i,\ x_\varepsilon^s(t,\ \theta))-(\nabla H_i,\ x_\varepsilon^u(t,\ \theta))\equiv D_i^s(t,\ \theta,\ \varepsilon)-D_\varepsilon^u(t,\theta,\varepsilon) \tag{3.3}$$

In turn, by virtue of Lemma 1 we have

$$D^\varkappa{}_i(t,\theta,\varepsilon)=(\nabla H_i,x_0(t,\theta))+\sum_{|n|=1}^{k}\varepsilon^n(\nabla H_i,\xi_n^\varkappa(t,\theta))+ \tag{3.4}$$
$$(\nabla H_i,a_\varkappa(t,\theta,\varepsilon))=\Delta_{i0}(t,\theta)+\sum_{|n|=1}^{k}\varepsilon^n\Delta_{in}^\varkappa(t,\theta)+\delta_\varkappa(t,\theta,\varepsilon)$$

For $|n|\geqslant 1$ we have

$$\frac{d}{dt}\Delta_{in}^\varkappa(t,\theta)=(d^2H_i\cdot I\nabla H_1,\xi_n^\varkappa(t,\theta))+(\nabla H_i,Id^2H_1\xi_n^\varkappa(t,\theta)+$$
$$P_n^\varkappa)=(d^2H_i\cdot I\nabla H_1,\xi_n^\varkappa(t,\theta))+(\nabla H_i,Id^2H_1\xi_n^\varkappa(t,\theta))+(\nabla H_i,P_n^\varkappa)$$

and for $i=1$ we obtain

$$\frac{d}{dt}\Delta_{1n}^\varkappa(t,\theta)=(d^2H_1\cdot I\nabla H_1,\xi_n^\varkappa(t,\theta))-(d^2H_1\cdot I\nabla H_1,\xi_n^\varkappa(t,\theta))+(\nabla H_1,P_n^\varkappa)=(\nabla H_1,P_n^\varkappa)$$

where we utilized the fact that the matrix d^2H_1 is symmetric and matrix I is skew symmetric. For $i=2$ we have

$$\frac{d}{dt}\Delta_{2n}^\varkappa(t,\theta)=(d^2H_2\cdot I\nabla H_1,\xi_n^\varkappa(t,\theta))+(\nabla H_2,Id^2H_1\xi_n^\varkappa(t,\theta))+(\nabla H_2,P_n^\varkappa)$$

Let us make use of the identity obtained by differentiating in x the identities $\{H_1,\ H_2\}\equiv(I\nabla H_1(x),\ \nabla H_2(x))\equiv 0$. Then

$$\frac{d}{dt}\Delta_{2n}^\varkappa(t,\theta)=(\nabla H_2,P_n^\varkappa)$$

Integrating the resulting equations for $\Delta_{in}^s(t,\ \theta)$ in t from 0 to $+\infty$ and using conditions 2) of Lemma 1, we obtain

$$\Delta_{in}^s(0,\theta)=-\int_0^\infty(\nabla H_i,P_n^s)\,dt$$

and similarly

$$\Delta_{in}^u(0,\theta)=\int_{-\infty}^0(\nabla H_i,P_n^u)\,dt$$

We note that

$$P_{10}^s=P_{10}^u=\frac{\partial p}{\partial\varepsilon_1}(x_0(t,\theta),0),\quad P_{01}^s=P_{01}^u=\frac{\partial p}{\partial\varepsilon_2}(x_0(t,\theta),0)$$

Subsituting the resulting expressions into (3.3), taking (3.4) into account and setting $t=0$, we obtain

$$\Delta_i(\theta,\varepsilon)=-\varepsilon_1d_{i1}-\varepsilon_2d_{i2}-\sum_{|n|=2}^{K}\varepsilon^n\Big[\int_0^\infty(\nabla H_i,P_n^s)\,dt+$$

$$\int_{-\infty}^{0} (\nabla H_i, P_n{}^u)\, dt\Big] + (\nabla H_i, a_s(0,\theta,\varepsilon) - a_u(0,\theta,\varepsilon))$$

$$\left(d_{ij}(\theta) = \int_{-\infty}^{\infty} \left(\nabla H_i, \frac{\partial p}{\partial \varepsilon_j}(x_0(t,\theta),0)\right) dt,\ i,j = 1,2\right)$$

Theorem 1. Let

$$B(\theta) = d_{11}d_{22} - d_{12}d_{21} = 0,\ \Sigma\, d_{ij}{}^2(\theta_*) \neq 0,\ B'(\theta_*) \neq 0$$

at $\theta = \theta_*$. Then a neighborhood of the point $(\theta_*, 0, 0) \in [0, 2\pi] \times R^2$ exists in which the solution of the system of equations $\Delta_1(\theta, \varepsilon) = 0$, $\Delta_2(\theta, \varepsilon) = 0$ is a smooth curve of the form $\varepsilon_2 = f_1(\varepsilon_1)$, $\theta = g_1(\varepsilon_1)$ or $\varepsilon_1 = f_2(\varepsilon_2)$, $\theta = g_2(\varepsilon_2)$ $(f_i(0) = g_i(0) - \theta_* = 0,\ i = 1, 2)$.

Proof. We assume for definiteness that $d_{12}(\theta_*) \neq 0$. The system has a curve of solutions $\varepsilon_1 = 0$, $\varepsilon_2 = 0$, therefore point $(\theta_*, 0, 0)$ is a solution of the system. By virtue of the theorem on implicit functions there exists a neighborhood of this point in which the first equation has a solution $\varepsilon_2 = h(\theta, \varepsilon_1)$. We substitute this solution into the second equation to obtain $\Delta_2(\theta, \varepsilon_1, h(\theta,\varepsilon_1)) \equiv \varphi(\theta, \varepsilon_1) = 0$. The point $(\theta_*, 0)$ is a nondegenerate critical saddle point of the function $\varphi(\theta, \varepsilon_1)$. Indeed, at the point $(\theta_*, 0)$

$$\frac{\partial \varphi}{\partial \theta} = \frac{\partial \Delta_2}{\partial \theta} - \frac{\partial \Delta_2}{\partial \varepsilon_2}\frac{\partial \Delta_1}{\partial \theta}\left(\frac{\partial \Delta_1}{\partial \varepsilon_2}\right)^{-1} = 0, \quad \frac{\partial \varphi}{\partial \varepsilon_1} = B(\theta_*)\, d_{12}^{-1}(\theta_*) = 0$$

since $\Delta_i(\theta, 0, 0) \equiv 0$ $(i = 1,2)$.

The determinant of the matrix of the second derivatives of the function $\varphi(\theta, \varepsilon_1)$ at the point $(\theta_*, 0)$ is equal to $[B'(\theta_*)\, d_{12}{}^{-1}(\theta_*)]^2 < 0$. Since the equation $\varphi(\theta, \varepsilon_1) = 0$ has a solution $\varepsilon_1 = 0$, it follows that in some neighborhood of the point $(\theta_*, 0)$ we also have the solution $\theta = g_1(\varepsilon_1)$, $g_1(0) = \theta_*$. For the initial system of equations we obtain the solution $\varepsilon_2 = h(g_1(\varepsilon_1), \varepsilon_1) = f_1(\varepsilon_1)$, $\theta = g_1(\varepsilon_1)$, and this completes the proof of the theorem.

Corollary. A neighborhood of the point $(0, 0)$ exists in the $(\varepsilon_1, \varepsilon_2)$-parameter space such, that every zero of the function $B(\theta)$ corresponding to conditions of Theorem 1 has a corresponding smooth bifurcation curve passing through the point $(0, 0)$ and corresponding to the loop of the separatrix of the singularity, intersecting the secant N at a unique point ("single circuit" loop).

Note. By virtue of the results of /2/ an enumerable set of "multicircuit" loops, i.e. loops intersecting N more than once, exists in the neighborhood of the single circuit loop of separatrix.

4. Let us consider the system (2.1) with Hamiltonian perturbation, i.e. $p(x, \varepsilon) = I\nabla K(x, \varepsilon)$, $\varepsilon \in R^1$. As in Sect.3, we introduce the functions $\Delta(\theta, \varepsilon)$ and $\Delta_2(\theta, \varepsilon)$ which, in the case of a one-parameter family of perturbations, have the form

$$\Delta_i(\theta,\varepsilon) = -\varepsilon \int_{-\infty}^{\infty} \left(\nabla H_i, I\nabla \frac{\partial K}{\partial \varepsilon}(x_0(t,\theta),0)\right) dt -$$
$$\sum_{n=2}^{k} \varepsilon^n \left(\int_0^{\infty} (\nabla H_i, P_n{}^s)\, dt + \int_{-\infty}^{0} (\nabla H_i, P_n{}^u)\, dt\right) +$$
$$(\nabla H_i, a_s(0,\theta,\varepsilon) - a_u(0,\theta,\varepsilon)) = -\varepsilon d_i(\theta) + \ldots$$

Lemma 2. There exists $\varepsilon_0 > 0$ such that $\Delta_2(\theta, \varepsilon) = 0$ when $|\varepsilon| < \varepsilon_0$ if and only if $x_\varepsilon{}^s(\theta) = x_\varepsilon{}^u(\theta)$.

Proof. In the course of proving the lemma we write $\nabla H_i(x_0(\theta)) = \nabla H_i$ $(i = 1, 2)$. The sufficiency of the conditions of the lemma is obvious. To prove the necessity we assume that $x_\varepsilon{}^s(\theta) \neq x_\varepsilon{}^u(\theta)$ for some fixed θ and ε (the quantity ε shall be chosen below). We introduce, on the two-dimensional disc N_θ, the coordinates given by the formula $x = x_0(\theta) + \xi\nabla H_1 + \eta\nabla H_2$. The points $x_\varepsilon{}^s(\theta)$ and $x_\varepsilon{}^u(\theta)$ lie on the curve described by the equation

$$R(\xi,\eta,\varepsilon) \equiv H_1(x_0(\theta) + \xi\nabla H_1 + \eta\nabla H_2) + K(x_0(\theta) + \xi\nabla H_1 + \eta\nabla H_2, \varepsilon) - H_1(0) - K(0, \varepsilon)$$

We can apply the theorem on implicit functions to this equation, since $R(0, 0, 0) = 0$, as $R_\xi'(0, 0, 0) = (\nabla H_1)^2 \neq 0$. Solving the equation for ξ, we obtain $\xi = \Phi(\eta, \varepsilon)$. Using the coordinates defined above, we can write $x_\varepsilon{}^s(\theta)$ and $x_\varepsilon{}^u(\theta)$ in the form

$$x_\varepsilon{}^s(\theta) = x_0(\theta) + \xi_s\nabla H_1 + \eta_s\nabla H_2,\quad x_\varepsilon{}^u(\theta) = x_0(\theta) + \xi_u\nabla H_1 + \eta_u\nabla H_2$$

therefore we have

$$x_\varepsilon{}^s(\theta) - x_\varepsilon{}^u(\theta) = (\xi_s - \xi_u)\nabla H_1 + (\eta_s - \eta_u)\nabla H_2 =$$
$$(\Phi(\eta_s, \varepsilon) - \Phi(\eta_u, \varepsilon))\nabla H_1 + (\eta_s - \eta_u)\nabla H_2 =$$
$$(\Phi_\eta'(\eta_*, \varepsilon)\nabla H_1 + \nabla H_2)(\eta_s - \eta_u)$$

The derivative $\Phi_\eta'(0, 0) = -(\nabla H_1, \nabla H_2)/(\nabla H_1)^2 \neq 0$, therefore we have

$$x_\varepsilon^s(\theta) - x_\varepsilon^u(\theta) = \left[-\left(\frac{(\nabla H_1, \nabla H_2)}{(\nabla H_1)^2} + \rho(\eta_*, \varepsilon)\right)\nabla H_1 + \nabla H_2\right](\eta_s - \eta_u)$$

where $\rho(\eta_*, \varepsilon) \to 0$ when $|\eta_*| + |\varepsilon| \to 0$. By definition we have $x_\varepsilon{}^s(\theta) \neq x_\varepsilon{}^u(\theta)$, therefore $\eta_s - \eta_u \neq 0$. Scalar multiplying both sides of the last equation by ∇H_2, we obtain

$$\Delta_2(\theta, \varepsilon) = (\eta_s - \eta_u)\left[\frac{\Gamma(\nabla H_1, \nabla H_2)}{(\nabla H_1)^2} + \rho(\eta_*, \varepsilon)(\nabla H_1, \nabla H_2)\right]$$

where $\Gamma(\nabla H_1, \nabla H_2)$ is the Gramm determinant (different from zero, since the vectors ∇H_1 and ∇H_2 are linearly independent); $\eta_* \to 0$ as $\varepsilon \to 0$, therefore $\Delta_2 \neq 0$ at sufficiently small ε which contradicts the initial assumption. The lemma implies that in the case of Hamiltonian perturbations the splitting of the separatrix surfaces is determined by the zeros of a single function $\Delta_2(\theta_1, \varepsilon)$.

Theorem 2. Let $d_2(\theta_*) = 0$, $d_2'(\theta_*) \neq 0$. Then a neighborhood of the point $(\theta_*, 0) \in [0, 2\pi] \times R^1$ exists, in which the equation $\Delta_2(\theta, \varepsilon) = 0$ has a unique solution $\theta = f(\varepsilon)$, $f(0) = \theta_*$.

Proof. The point $(\theta_*, 0)$ is a nondegenerate critical saddle point of the function $\Delta_2(\theta, \varepsilon)$, since at this point

$$\Delta_2 = \frac{\partial \Delta_2}{\partial \theta} = \frac{\partial \Delta_2}{\partial \varepsilon} = \frac{\partial^2 \Delta_2}{\partial \theta^2} = 0, \quad \frac{\partial^2 \Delta_2}{\partial \theta \partial \varepsilon} = d_2'(\theta_*) \neq 0$$

Since the equation $\Delta_2(\theta, \varepsilon) = 0$ has a solution $\varepsilon = 0$, therefore it also has the solution $\theta = f(\varepsilon)$, $f(0) = \theta_*$.

Note. It can be shown that on the set of the level Σ_ε the manifolds $W_\varepsilon{}^s$ and $W_\varepsilon{}^u$ intersect transversally along the loop of a separatrix existing, according to Theorem 2, when $\varepsilon \neq 0$.

5. We give an example of an integrable Hamiltonian system satisfying the assumptions of this paper. The Hamiltonian H_1 and first integral H_2 are given by the expressions

$$H_1 = \alpha(p_1 q_1 + p_2 q_2) - \beta(p_1 q_2 - p_2 q_1) + A[(p_1^2 + p_2^2)^2 + (q_1^2 + q_2^2)^2]$$
$$H_2 = p_1 q_2 - p_2 q_1;\ \alpha < 0,\ \beta > 0,\ A = -\alpha/\sqrt{2}$$

The point $O(0, 0, 0, 0)$ denotes a state of equilibrium of saddle-focus type, and the quantity $\tau = \alpha \neq 0$. A $C_0 > 0$ exists such that when $C > C_0$, the manifold $H_1 = C$ is homeomorphic to a three-dimensional spherical shell, and the sphere bounded by this shell contains all levels of the function H_1 with smaller values of C. From this it follows that all levels of the function H_1 are compact and the trajectories of the vector fields $I\nabla H_i$ $(i = 1,2)$ are therefore defined for all $t \in R^1$.

We shall show that a trajectory exists which is doubly asymptotic to the point O. Let us perform the (noncanonical) coordinate change $p_1 = r\cos\varphi$, $p_2 = r\sin\varphi$, $q_1 = \rho\cos\theta$, $q_2 = \rho\sin\theta$. The initial system and the integrals H_1 and H_2 can be written in terms of these coordinates in the form

$$r^\cdot = -\alpha r - 4A\rho^3\cos(\varphi - \theta), \quad \rho^\cdot = \alpha\rho + 4Ar^3\cos(\varphi - \theta) \tag{5.1}$$
$$r\varphi^\cdot = \beta r + 4A\rho^3\sin(\varphi - \theta), \quad \rho\theta^\cdot = \beta\rho + 4Ar^3\sin(\varphi - \theta) \tag{5.2}$$
$$H_1 = \alpha r\rho\cos(\varphi - \theta) + \beta r\rho\sin(\varphi - \theta) + A(r^4 + \rho^4)$$
$$H_2 = -r\rho\sin(\varphi - \theta)$$

On the separatrix manifolds we have $H_1 = 0$ and $H_2 = 0$. The set of solutions of the system $H_1 = 0$, $H_2 = 0$ is determined by the conditions $\varphi = \theta$, $h \equiv \alpha\rho r + A(\rho^4 + r^4) = 0$, since $\rho > 0$, $r > 0$, $\alpha A < 0$. Under these conditions (5.2) yields $\varphi(t) = \beta t + \theta^\circ$, $\theta(t) = \beta t + \theta^\circ$. Carrying out in (5.1) the substitution $u = \rho^4 + r^4$, $v = \rho^4 - r^4$, $s = -4\alpha t$ and remembering that on the separatrix loop $h = 0$, we obtain the following expression in the variables u, v, s: $u^\cdot = -v$, $v^\cdot = -u + 2u^3$. Thus we have $u = \mathrm{ch}^{-1}s$, $v = \mathrm{sh}\, s\, \mathrm{ch}^{-2}s$ for the loop. Returning now to the initial system, we obtain the following one-parameter family of doubly asymptotic solutions:

$$p_j(t, \theta) = e^{\alpha t}A_j, \quad q_j = e^{-\alpha t}A_j, \quad j = 1, 2$$
$$(A_1 = 2^{-1/4}(\mathrm{ch}\, 4\alpha t)^{-1/2}\cos(\beta t + \theta);\ A_2 = 2^{-1/4}(\mathrm{ch}\, 4\alpha t)^{-1/2}\sin(\beta t + \theta))$$

To illustrate Theorem 1 and 2, we shall quote the results of computing the functions $B(\theta)$ and $d_2(\theta)$ for the concrete perturbations. In the first case $p(x, \varepsilon) = (\varepsilon_1 p_2, -\varepsilon_2 p_1, (\varepsilon_1 + \varepsilon_2) q_2, 0)$ and

$$B(\theta) = -\frac{\pi^2 \sin 4\theta}{64\alpha}\left(\mathrm{ch}\ \frac{2\beta\pi}{4\alpha}\right)^{-2}$$

and in the second case $K(x, \varepsilon) = \varepsilon(p_1^2 + q_1^2)$, while

$$d_2(\theta) = \frac{\pi}{4\alpha}\ \sin 2\theta\, \mathrm{ch}\frac{\pi\beta}{4\alpha}\left(\mathrm{ch}\frac{\pi\beta}{2\alpha}\right)^{-1}$$

6. The results obtained in Sect.1–5 can be generalized to the case when the initial system (1.1) has two nondegenerate singularities O_1 and O_2 of saddle-focus type, and a trajectory connecting them. We shall assume for definiteness that the trajectory belongs to the intersection of the stable manifold $W_0^s(O_1)$ with the unstable manifold $W_0^u(O_2)$. As in Assertion 1, we can show that in this case $W_0^s(O_1) - \{O_1\}$ and $W_0^u(O_2) - \{O_2\}$ coincide. If we demand that the singularities O_1 and O_2 remain in the same place in the system (2.1) for all ε, and that in the case of Hamiltonian perturbations both these points also lie on the same level $H_1 + K = \mathrm{const}$, then Theorems 1 and 2 can be applied totally to this case. It should be noted that $x_\varepsilon^s(t, \theta)$ in the expressions (3.1) refer to the point O_1 and $x_\varepsilon^u(t, \theta)$ to O_2.

Analogous results are obtained for the perturbations of an integrable system with a doubly asymptotic trajectory towards the saddle type singularity (the eigenvalues $\pm\lambda_1, \pm\lambda_2$, $\lambda_1 \neq \lambda_2$), along which $I\nabla H_1$ and $I\nabla H_2$ are independent. In addition, the results of this paper can be extended to the case of nonautonomous, time-periodic perturbations $p(x, \varepsilon, t)$.

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